

On the Lattice Packings and Coverings of the Plane with Convex Quadrilaterals

Ktrati Sriamorn

December 22, 2014

Abstract

It is well known that the lattice packing density and the lattice covering density of a triangle are $\frac{2}{3}$ and $\frac{3}{2}$ respectively [3]. We also know that the lattices that attain these densities both are unique. Let $\delta_L(K)$ and $\vartheta_L(K)$ denote the lattice packing density and the lattice covering density of K , respectively. In this paper, I study the lattice packings and coverings for a special class of convex disks, which includes all triangles and convex quadrilaterals. In particular, I determine the densities $\delta_L(Q)$ and $\vartheta_L(Q)$, where Q is an arbitrary convex quadrilateral. Furthermore, I also obtain all of lattices that attain these densities. Finally, I show that $\delta_L(Q)\vartheta_L(Q) \geq 1$ and $\frac{1}{\delta_L(Q)} + \frac{1}{\vartheta_L(Q)} \geq 2$, for each convex quadrilateral Q .

1 Introduction and preliminaries

An n -dimensional *convex body* is a compact convex subset of \mathbb{R}^n with an interior point. A 2-dimensional convex body is called a *convex disk*. The n -dimension measure of a set S will be denoted by $|S|$. The closure and the interior of S will be denoted by \overline{S} and $Int(S)$, respectively. The cardinality of S is denoted by $card\{S\}$.

For any n independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in \mathbb{R}^n , the *lattice* generated by $\mathbf{v}_1, \dots, \mathbf{v}_n$ is the set of vectors

$$\{k_1\mathbf{v}_1 + \dots + k_n\mathbf{v}_n : k_1, \dots, k_n \text{ are integers}\},$$

denoted by $\mathcal{L}(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

A family $\mathcal{F} = \{K_1, K_2, \dots\}$ of convex bodies is called a *covering* of a domain $D \subseteq \mathbb{R}^n$ provided $D \subseteq \bigcup_i K_i$, and call \mathcal{F} a *packing* of D if the interiors are mutually disjoint and $\bigcup_i K_i \subseteq D$. A family \mathcal{F} which is both a packing and a covering of D is called a *tiling* of D . Any compact set which admits a tiling of \mathbb{R}^n is called a *tile*.

The *upper* and *lower density* of a family $\mathcal{F} = \{K_1, K_2, \dots\}$ of convex bodies with respect to a bounded domain D are defined as

$$d_+(\mathcal{F}, D) = \frac{1}{|D|} \sum_{K \in \mathcal{F}, K \cap D \neq \emptyset} |K|,$$

and

$$d_-(\mathcal{F}, D) = \frac{1}{|D|} \sum_{K \in \mathcal{F}, K \subset D} |K|.$$

We define the *upper* and *lower density* of the family \mathcal{F} by

$$d_+(\mathcal{F}) = \limsup_{r \rightarrow \infty} d_+(\mathcal{F}, rB^n),$$

and

$$d_-(\mathcal{F}) = \liminf_{r \rightarrow \infty} d_-(\mathcal{F}, rB^n),$$

where B^n denotes the unit ball in \mathbb{R}^2 , centered at the origin.

The *packing density* $\delta(K)$ of convex body K is defined by the formula

$$\delta(K) = \sup_{\mathcal{F}} d_+(\mathcal{F}),$$

the supremum being taken over all packings \mathcal{F} of \mathbb{R}^n with congruent copies of K . The *covering density* $\vartheta(K)$ of convex body K is defined by the formula

$$\vartheta(K) = \inf_{\mathcal{F}} d_-(\mathcal{F}),$$

the infimum being taken over all coverings \mathcal{F} of \mathbb{R}^n with congruent copies of K .

One may consider arrangements of translated copies of K only, or just lattice arrangements of translates of K . In these cases, the corresponding densities assigned to K by analogous definitions are: the *translative* packing and covering density of K , denoted by $\delta_T(K)$ and $\vartheta_T(K)$, and the *lattice* packing and covering density of K , denoted by $\delta_L(K)$ and $\vartheta_L(K)$, respectively.

Let K be an n -dimensional convex body. Suppose that $x \in \mathbb{R}^n$ and X is a discrete subset of \mathbb{R}^n , we define $K + x = \{y + x : y \in K\}$, and denote by $K + X$ the family $\{K + x\}_{x \in X}$.

An *optimal packing lattice* of K is a lattice Λ which $K + \Lambda$ is a packing of \mathbb{R}^n with density $\delta_L(K)$. Denote by $\Delta(K)$ the collection of all optimal packing lattice of K . Similarly, An *optimal covering lattice* of K is a lattice Λ which $K + \Lambda$ is a covering of \mathbb{R}^n with density $\vartheta_L(K)$. Denote by $\Theta(K)$ the collection of all optimal covering lattice of K .

Let \mathcal{K}^n denote the collection of all n -dimensional convex bodies. Let $K_1 + K_2$ denote the *Minkowski sum* of K_1 and K_2 defined by

$$K_1 + K_2 = \{x_1 + x_2 : x_i \in K_i\},$$

let $\|\cdot\|^*$ denote the *Hausdorff metric* on \mathcal{K}^n defined by

$$\|K_1, K_2\|^* = \min\{r : K_1 \subset K_2 + rB^n, K_2 \subset K_1 + rB^n\},$$

and let $\{\mathcal{K}^n, \|\cdot\|^*\}$ denote the space of \mathcal{K}^n with metric $\|\cdot\|^*$. It is easy to see that, for $\lambda_i \in \mathbb{R}$ and $K_i \in \mathcal{K}^n$,

$$\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m \in \mathcal{K}^n.$$

In certain sense, the space \mathcal{K}^n has linear structure. Blaschke selection theorem guarantees the local compactness of $\{\mathcal{K}^n, \|\cdot\|^*\}$. It is easy to show that all

$\delta(K), \delta_T(K), \delta_L(K), \vartheta(K), \vartheta_T(K)$ and $\vartheta_L(K)$ are bounded continuous functions defined on $\{\mathcal{K}^n, \|\cdot\|^*\}$.

Define $\omega : \mathcal{K}^2 \rightarrow \mathbb{R}^2$ by $\omega(K) = (\delta(K), \vartheta(K))$ for every $K \in \mathcal{K}^2$. By continuity of each of the real-valued functions δ and ϑ , the function ω is continuous. Let $\Omega = \omega(\mathcal{K}^2)$. Similarly, we can define the sets Ω_T and Ω_L by replacing the function $\omega = (\delta, \vartheta)$ by $\omega_T = (\delta_T, \vartheta_T)$ and by $\omega_L = (\delta_L, \vartheta_L)$, respectively.

Since the Minkowski sum of two *centrally symmetric* sets is a centrally symmetric set, the analogous statements hold for the space \mathcal{C}^n of centrally symmetric n -dimensional convex bodies, and to the corresponding sets $\Omega^* = \omega(\mathcal{C}^2)$, $\Omega_T^* = \omega_T(\mathcal{C}^2)$ and $\Omega_L^* = \omega_L(\mathcal{C}^2)$. It is well known that $\vartheta_T(C) = \vartheta_L(C)$, for all centrally symmetric convex disks K [4]. This immediately follows that $\Omega_T^* = \Omega_L^*$.

The question of describing explicitly the sets $\Omega, \Omega_T, \Omega_L, \Omega^*$, and Ω_L^* remains open. In 2001 Ismailescu [7] proved that for each centrally symmetric convex disk C ,

$$1 - \delta_L(C) \leq \vartheta_L(C) - 1 \leq \frac{5}{4} \sqrt{1 - \delta_L(C)}.$$

These inequalities can be expressed as : the set $\Omega_L^* = \Omega_T^*$ lies between the line $x + y = 2$ and the curve $y = 1 + \frac{5}{4} \sqrt{1 - x}$. A recent paper of Ismailescu and Kim [8] showed that $\delta_L(C)\vartheta_L(C) \geq 1$ for every centrally symmetric convex disk K , which is stronger than Ismailescu's inequality $\delta_L(C) + \vartheta_L(C) \geq 2$ mentioned above. It is still unknown whether these inequalities hold for any (non-symmetric) convex disks. However, I will show later that these inequalities hold for convex quadrilaterals.

We note that if one can prove that the inequality $\frac{1}{\delta_L(C)} + \frac{1}{\vartheta_L(C)} \leq 2$ holds for every *centrally symmetric* convex disk C , this would represent an improvement over the inequality $\delta_L(C)\vartheta_L(C) \geq 1$. Unfortunately, this still is an open problem. In general case (not necessarily symmetrical), it is obvious that there exist convex disks K such that the inequality $\frac{1}{\delta_L(K)} + \frac{1}{\vartheta_L(K)} \leq 2$ does not hold. In fact, $\frac{1}{\delta_L(T)} + \frac{1}{\vartheta_L(T)} = \frac{3}{2} + \frac{2}{3} > 2$, for triangles T . Moreover, we will see later in this paper that $\frac{1}{\delta_L(Q)} + \frac{1}{\vartheta_L(Q)} \geq 2$, for every convex quadrilaterals Q .

Let $f(x)$ be a convex and non-increasing continuous function with $f(0) = 1$ and $f(1) \geq 0$. Throughout this paper, we define the convex disk

$$K_f = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq f(x)\}.$$

Denote by C_f the curve $\{(x, f(x)) : 0 < x < 1\} \cup \{(1, y) : 0 < y \leq f(1)\}$.

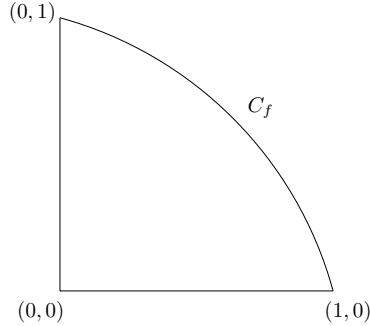


Figure 1: K_f

A very recent paper of Xue and Kirati [10] proved the following result

Theorem 1.1. $\vartheta_T(K_f) = \vartheta_L(K_f)$.

2 Main Results

Let $K_{x,y}$ denote the quadrilateral with vertices $(0,1)$, $(0,0)$, $(1,0)$ and (x,y) . Denote by D the set of all points (x,y) that satisfy $0 \leq x \leq y \leq 1$ and $x+y \geq 1$. Let Q be an arbitrary convex quadrilateral. One can easily show that there exists $(x,y) \in D$ such that Q and $K_{x,y}$ are affinely equivalent.

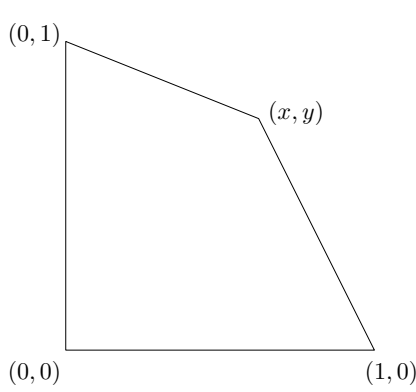


Figure 2: $K_{x,y}$

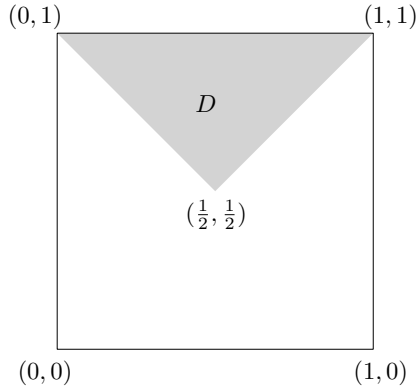


Figure 3: D

It is well known that δ_L and ϑ_L are affinely invariant. In order to determine $\delta_L(Q)$, $\vartheta_L(Q)$, $\Delta(Q)$ and $\Theta(Q)$, we may assume, without loss of generality, that $Q = K_{x,y}$, where $(x,y) \in D$. We define

$$\delta_L(x,y) = \delta_L(K_{x,y}), \quad \vartheta_L(x,y) = \vartheta_L(K_{x,y}),$$

and

$$\Delta(x,y) = \Delta(K_{x,y}), \quad \Theta(x,y) = \Theta(K_{x,y}).$$

The main results are as follows

Theorem 2.1. Suppose that $(x,y) \in D$, then

$$\delta_L(x,y) = \frac{2y(x+y)}{4y+x-1}$$

and

$$\vartheta_L(x,y) = \begin{cases} \frac{3(x+y)(1-x)}{2y} & x \leq \frac{1}{3}, \\ \frac{2(x+y)}{y(1+3x)} & x \geq \frac{1}{3} \text{ and } y \geq \frac{2}{3}, \\ \frac{(x+y)(4(1-x)(1-y)-xy)}{2(x(1-x)+y(1-y)-xy)} & y < \frac{2}{3}. \end{cases}$$

Theorem 2.2. Suppose that $(x,y) \in D$, then

$$\Delta(x,y) = \begin{cases} \{\mathcal{L}((0,1), (1,t)), \mathcal{L}((t,1), (1,0)) : t \in (0,1)\} & x=y=1, \\ \left\{ \mathcal{L}\left(\left(\frac{y-1}{2y}, 1\right), \left(\frac{3y-1}{2y}, \frac{1}{2}\right)\right), \right. \\ \quad \left. \mathcal{L}\left(\left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, \frac{3y-1}{2y}\right)\right) \right\} & \frac{1}{2} < x=y < 1, \\ \left\{ \mathcal{L}\left(\left(\frac{x-1}{2y}, 1\right), \left(1 - \frac{1-x}{2y}, \frac{1}{2}\right)\right) \right\} & \text{otherwise,} \end{cases}$$

and

$$\Theta(x, y) = \begin{cases} \left\{ \mathcal{L} \left(\left(-\frac{1}{3}, \frac{2y}{3(1-x)} \right), \left(\frac{1}{3}, \frac{y}{3(1-x)} \right) \right) \right\} & x \leq \frac{1}{3}, \\ \left\{ \mathcal{L} \left(\left(\frac{x-1}{2}, y \right), \left(x, \frac{y}{2} \right) \right) \right\} & \frac{1}{3} < x < y, \frac{2}{3} \leq y, \\ \left\{ \mathcal{L} \left(\left(\frac{y(1+y-2x)-2(1-x)^2}{4(1-x)(1-y)-xy}, \frac{(2-y)(1-x)-y^2}{4(1-x)(1-y)-xy} \right), \right. \right. \\ \quad \left. \left(\frac{x(2(1-x)-y)}{4(1-x)(1-y)-xy}, \frac{y(2(1-y)-x)}{4(1-x)(1-y)-xy} \right) \right) \right\} & y < \frac{2}{3}, \\ \left\{ \mathcal{L} \left(\left(\frac{y-1}{2}, y \right), \left(y, \frac{y}{2} \right) \right), \mathcal{L} \left(\left(-\frac{y}{2}, \frac{1+y}{2} \right), \left(\frac{y}{2}, y \right) \right) \right\} & \frac{2}{3} < x = y < 1, \\ \left\{ \mathcal{L} \left(\left(-\frac{t}{2}, \frac{1+t}{2} \right), (1-t, t) \right) : t \in \left[\frac{1}{3}, \frac{2}{3} \right] \right\} & x = y = \frac{2}{3}, \\ \left\{ \mathcal{L}((0, 1), (1, t)), \mathcal{L}((t, 1), (1, 0)) : t \in (0, 1] \right\} & x = y = 1. \end{cases}$$

Corollary 2.3. Suppose that $(x, y) \in D$, then

$$\text{card}\{\Delta(x, y)\} = \begin{cases} \mathfrak{c} & x = y = 1 \\ 2 & \frac{1}{2} < x = y < 1, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\text{card}\{\Theta(x, y)\} = \begin{cases} \mathfrak{c} & x = y = \frac{2}{3} \text{ or } x = y = 1, \\ 2 & \frac{2}{3} < x = y < 1, \\ 1 & \text{otherwise,} \end{cases}$$

where \mathfrak{c} is the cardinality of the real numbers.

Corollary 2.4. For every convex quadrilaterals Q , we have

$$\delta_L(Q)\vartheta_L(Q) \geq 1$$

and

$$\frac{1}{\delta_L(Q)} + \frac{1}{\vartheta_L(Q)} \geq 2.$$

Corollary 2.5. For every convex quadrilaterals Q , we have

$$\delta_L(Q) + \vartheta_L(Q) \geq 2$$

and

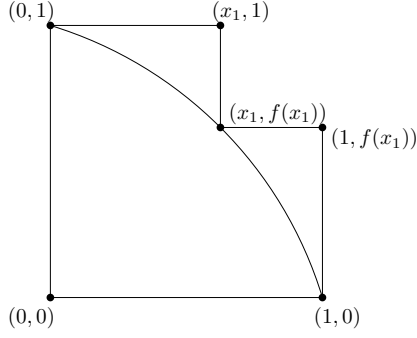
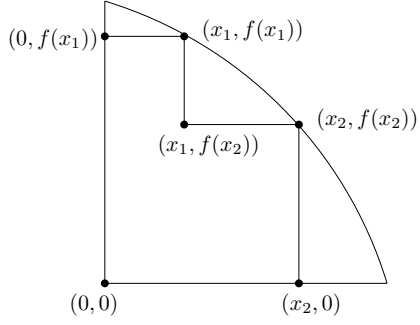
$$\vartheta_L(Q) \leq 1 + \frac{5}{4}\sqrt{1 - \delta_L(Q)}.$$

Remark 2.1. It is well known that $\delta_T(K) = \delta_L(K)$, for every convex disks K (see [2] or [5]). Furthermore, according to Theorem 1.1, we know that $\vartheta_T(Q) = \vartheta_L(Q)$, for each convex quadrilateral Q . Therefore, the above statements also hold for δ_T and ϑ_T .

3 The lattice packings and coverings of K_f

In this section, for convenience, we assume that K_f is not a unit square.

For any real numbers x, x', y, y' , let $L(x, y, x', y')$ denote the line segment between (x, y) and (x', y') . Suppose that $0 \leq x \leq 1$, denote by $S^f(x)$ the polygon bounded by $L(0, 0, 0, 1)$, $L(0, 1, x, 1)$, $L(x, 1, x, f(x))$, $L(x, f(x), 1, f(x))$,

Figure 4: $S^f(x_1)$ Figure 5: $S_f(x_1, x_2)$

$L(1, f(x), 1, 0)$ and $L(1, 0, 0, 0)$. For $0 \leq x_1 \leq x_2 \leq 1$, denote by $S_f(x_1, x_2)$ the polygon bounded by $L(0, 0, 0, f(x_1))$, $L(0, f(x_1), x_1, f(x_1))$, $L(x_1, f(x_1), x_1, f(x_2))$, $L(x_1, f(x_2), x_2, f(x_2))$, $L(x_2, f(x_2), x_2, 0)$ and $L(x_2, 0, 0, 0)$.

Define

$$A^f = \min\{|S^f(x)| : 0 \leq x \leq 1\},$$

and

$$A_f = \max\{|S_f(x_1, x_2)| : 0 \leq x_1 \leq x_2 \leq 1\}.$$

Let

$$\mathcal{X}^f = \{x : 0 \leq x \leq 1, |S^f(x)| = A^f\},$$

and

$$\mathcal{X}_f = \{(x_1, x_2) : 0 \leq x_1 \leq x_2 \leq 1, |S_f(x_1, x_2)| = A_f\}.$$

The paper of Xue and Kirati [10] showed that

Theorem 3.1. $\vartheta_T(K_f) = \vartheta_L(K_f) = \frac{|K_f|}{A_f}$.

Furthermore, from the results of the paper, we can also deduce that

Lemma 3.2. *For any (2-dimension) lattice Λ , if $\Lambda \in \Theta(K_f)$ (i.e., $K_f + \Lambda$ is a covering of \mathbb{R}^2 with density $\vartheta_L(K_f)$), then there exist $0 \leq x_1 \leq x_2 \leq 1$ such that $S_f(x_1, x_2) \subset K_f$, $S_f(x_1, x_2) + \Lambda$ is a tiling of \mathbb{R}^2 , and $|S_f(x_1, x_2)| = A_f$*

For any $0 \leq x_1 \leq x_2 \leq 1$, it is easy to see that $S_f(x_1, x_2) + \mathcal{L}((x_1 - x_2, f(x_1)), (x_1, f(x_2)))$ is a tiling of \mathbb{R}^2 (see Figure 6). Define Λ_f by

$$\Lambda_f(x_1, x_2) = \mathcal{L}((x_1 - x_2, f(x_1)), (x_1, f(x_2))). \quad (1)$$

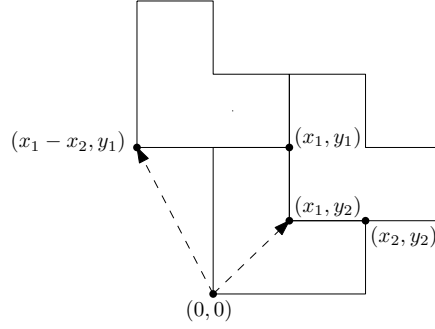
Since $S_f(x_1, x_2) \subset K_f$, we obtain $K_f + \Lambda_f(x_1, x_2)$ is a covering of \mathbb{R}^2 with density $\frac{|K_f|}{|S_f(x_1, x_2)|}$. By Theorem 3.1, we know that Λ_f can be seen as a mapping from \mathcal{X}_f to $\Theta(K_f)$. From Lemma 3.2, we obtain Λ_f is surjective. To show that Λ_f is injective, we first prove the following lemma.

Lemma 3.3. *Suppose that $(x_1, x_2) \in \mathcal{X}_f$. We have*

$$2x_1 \geq x_2, \quad 2x_2 \geq 1 + x_1,$$

and

$$2f(x_2) \geq f(x_1), \quad 2f(x_1) \geq 1 + f(x_2).$$

Figure 6: $\mathcal{L}((x_1 - x_2, y_1), (x_1, y_2))$

Proof. Since K_f is not a square, one can easily see that $0 < x_1 < x_2$. By the definition of \mathcal{X}_f , we know that for all sufficiently small $\epsilon > 0$,

$$\begin{aligned} x_1 f(x_1) + (x_2 - x_1) f(x_2) &= |S_f(x_1, x_2)| \\ &\geq |S_f(x_1 - \epsilon, x_2)| \\ &= (x_1 - \epsilon) f(x_1 - \epsilon) + (x_2 - x_1 + \epsilon) f(x_2). \end{aligned}$$

Since f is convex and $f(0) = 1$, we have

$$f(x_1 - \epsilon) \geq f(x_1) + \frac{1 - f(x_1)}{x_1} \epsilon.$$

Therefore

$$x_1 f(x_1) + (x_2 - x_1) f(x_2) \geq (x_1 - \epsilon) \left(f(x_1) + \frac{1 - f(x_1)}{x_1} \epsilon \right) + (x_2 - x_1 + \epsilon) f(x_2)$$

From this, we can easily obtain

$$2f(x_1) \geq 1 + f(x_2) - \frac{1 - f(x_1)}{x_1} \epsilon.$$

On allowing ϵ to tend to zero, we get

$$2f(x_1) \geq 1 + f(x_2).$$

By symmetry, we have

$$2x_2 \geq 1 + x_1.$$

On the other hand,

$$\begin{aligned} x_1 f(x_1) + (x_2 - x_1) f(x_2) &= |S_f(x_1, x_2)| \\ &\geq |S_f(x_1, x_2 - \epsilon)| \\ &= x_1 f(x_1) + (x_2 - \epsilon - x_1) f(x_2 - \epsilon). \end{aligned}$$

Because f is convex, we have

$$f(x_2 - \epsilon) \geq f(x_2) + \frac{f(x_1) - f(x_2)}{x_2 - x_1} \epsilon.$$

Therefore

$$x_1 f(x_1) + (x_2 - x_1) f(x_2) \geq x_1 f(x_1) + (x_2 - \epsilon - x_1) \left(f(x_2) + \frac{f(x_1) - f(x_2)}{x_2 - x_1} \epsilon \right).$$

From this, we can deduce that

$$2f(x_2) \geq f(x_1) - \frac{f(x_1) - f(x_2)}{x_2 - x_1} \epsilon,$$

and hence

$$2f(x_2) \geq f(x_1).$$

By symmetry, we immediately get

$$2x_1 \geq x_2.$$

This completes the proof. \square

Theorem 3.4. Λ_f is a bijection from \mathcal{X}_f to $\Theta(K_f)$

Proof. Let $(x_1, x_2), (x'_1, x'_2) \in \mathcal{X}_f$. Since K_f is not a square, one can show that

$$(x_1, f(x_2)) \in \text{Int}(K_f) \text{ and } (x'_1, f(x'_2)) \in \text{Int}(K_f).$$

By Lemma 3.3, we know that

$$x_1 + x_2 \geq 1, \quad f(x_1) + f(x_2) \geq 1$$

and

$$2x_1 \geq x_2, \quad 2f(x_2) \geq f(x_1).$$

This can be deduced that (see Figure 7)

$$\Lambda_f(x_1, x_2) \cap \text{Int}(K_f) = \{(x_1, f(x_2))\}.$$

By the same reason, we have

$$\Lambda_f(x'_1, x'_2) \cap \text{Int}(K_f) = \{(x'_1, f(x'_2))\}.$$

Now suppose that $(x'_1, x'_2) \neq (x_1, x_2)$. if $(x_1, f(x_2)) \neq (x'_1, f(x'_2))$, then we have $\Lambda_f(x_1, x_2) \neq \Lambda_f(x'_1, x'_2)$. if $(x_1, f(x_2)) = (x'_1, f(x'_2))$, then we may assume, without loss of generality, that $x_2 < x'_2$, i.e., $S_f(x_1, x_2) \subsetneq S_f(x'_1, x'_2)$, and hence $|S_f(x_1, x_2)| < |S_f(x'_1, x'_2)|$. This is impossible. It follows that Λ_f is an injection. \square

Now we determine $\delta_L(K_f)$. By the definition of $S^f(x)$, it is clear that $S^f(x) + \mathcal{L}((x-1, 1), (x, f(x)))$ is a tiling of \mathbb{R}^2 . Define Λ^f by

$$\Lambda^f(x) = \mathcal{L}((x-1, 1), (x, f(x))). \quad (2)$$

Since $K_f \subset S^f(x)$, the family $K_f + \Lambda^f(x)$ is a packing of \mathbb{R}^2 with density $\frac{|K_f|}{|S^f(x)|}$. We thus have

$$\delta_L(K_f) \geq \frac{|K_f|}{|S^f(x)|}.$$

Clearly, we have $K_{x,y} = K_{\bar{f}}$. Denote by I_1 and I_2 the intervals $[0, x]$ and $[x, 1]$, respectively. For $i, j \in \{1, 2\}$ and $i \leq j$, we define

$$A_{ij}(x, y) = \max_{x_1 \in I_i, x_2 \in I_j, x_1 \leq x_2} |S_{\bar{f}}(x_1, x_2)|.$$

Let $A_*(x, y) = \max\{A_{12}(x, y), A_{11}(x, y), A_{22}(x, y)\}$. By Theorem 3.1, we have

$$\vartheta_L(x, y) = \frac{|K_{x,y}|}{A_*(x, y)} = \frac{x+y}{2A_*(x, y)}. \quad (4)$$

In order to determine $\Theta(x, y)$, we define

$$\tilde{\mathcal{X}}_{ij}(x, y) = \{(x_1, x_2) : x_1 \in I_i, x_2 \in I_j, x_1 \leq x_2, |S_{\bar{f}}(x_1, x_2)| = A_{ij}(x, y)\},$$

and

$$\mathcal{X}_{ij}(x, y) = \{(x_1, x_2) : x_1 \in I_i, x_2 \in I_j, x_1 \leq x_2, |S_{\bar{f}}(x_1, x_2)| = A_*(x, y)\},$$

where $i, j \in \{1, 2\}$ and $i \leq j$. Obviously, we have

$$\mathcal{X}_{ij}(x, y) = \begin{cases} \tilde{\mathcal{X}}_{ij}(x, y) & A_{ij}(x, y) = A_*(x, y), \\ \emptyset & \text{otherwise.} \end{cases} \quad (5)$$

Let $\mathcal{X}_*(x, y) = \mathcal{X}_{12}(x, y) \cup \mathcal{X}_{11}(x, y) \cup \mathcal{X}_{22}(x, y)$. By Theorem 3.4 and (1), we know that

$$\Theta(x, y) = \{\mathcal{L}((x_1 - x_2, \bar{f}(x_1)), (x_1, \bar{f}(x_2))) : (x_1, x_2) \in \mathcal{X}_*(x, y)\} \quad (6)$$

4.1 $A_{12}(x, y)$ and $\tilde{\mathcal{X}}_{12}(x, y)$

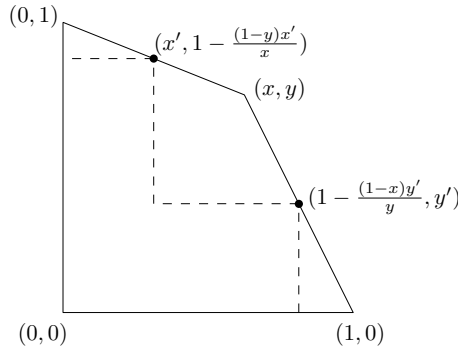


Figure 8: $S_{\bar{f}}\left(x', 1 - \frac{(1-x)y'}{y}\right)$

Let $G(x', y')$ denote the area of $S_{\bar{f}}(x', 1 - \frac{(1-x)y'}{y})$, where $0 \leq x' \leq x$ and $0 \leq y' \leq y$ (see Figure 8). Then we get

$$A_{12}(x, y) = \max_{0 \leq x' \leq x, 0 \leq y' \leq y} G(x', y').$$

By computations, we can obtain

$$G(x', y') = -\frac{(1-y)x'^2}{x} - \frac{(1-x)y'^2}{y} + x' + y' - x'y', \quad (7)$$

where $0 \leq x' \leq x$ and $0 \leq y' \leq y$. Obviously, G is a convex quadratic function with respect to x' (or y'). When y' is fixed, by considering the critical point, it is easy to see that $G(x', y')$ reaches its maximum value at

$$x' = \begin{cases} x & 0 \leq y' \leq 2y-1, \\ \frac{x(1-y')}{2(1-y)} & 2y-1 < y' \leq y. \end{cases} \quad (8)$$

Here we note that, when $y < 1$, we have $\frac{1-y'}{2(1-y)} \leq 1$ is equivalent to $2y-1 \leq y'$. By substituting (8) into (7), we obtain

$$g(y') = \begin{cases} -\frac{(1-x)y'^2}{y} + (1-x)y' + xy & 0 \leq y' \leq 2y-1, \\ \frac{1}{2} \left(\frac{x}{2(1-y)} - \frac{2(1-x)}{y} \right) y'^2 + \left(1 - \frac{x}{2(1-y)} \right) y' + \frac{x}{4(1-y)} & 2y-1 < y' \leq y, \end{cases} \quad (9)$$

and hence

$$A_{12}(x, y) = \max_{0 \leq y' \leq y} g(y').$$

The derivative of g with respect to y' is

$$\frac{dg}{dy'} = \begin{cases} -\frac{2(1-x)y'}{y} + (1-x) & 0 < y' < 2y-1, \\ \left(\frac{x}{2(1-y)} - \frac{2(1-x)}{y} \right) y' + 1 - \frac{x}{2(1-y)} & 2y-1 < y' < y. \end{cases} \quad (10)$$

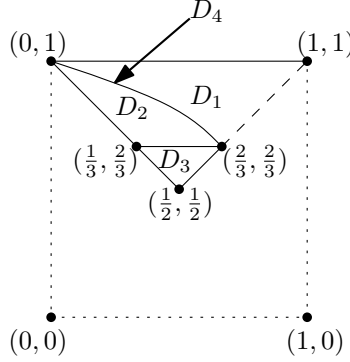
Remark 4.1. Obviously, the function g is a convex quadratic function on $[0, 2y-1]$, and if $\frac{y}{2} \leq 2y-1$ (i.e., $y \geq \frac{2}{3}$), then g reaches its maximum value on $[0, 2y-1]$ at $y' = \frac{y}{2}$.

Now we divide $D \setminus \{(1, 1)\}$ into six sets

$$\begin{aligned} D_1 &= \{(x, y) \in D : 4(1-x)(1-y) - xy < 0, x < y\}, \\ D_2 &= \{(x, y) \in D : y \geq \frac{2}{3}, 4(1-x)(1-y) - xy > 0\}, \\ D_3 &= \{(x, y) \in D : y < \frac{2}{3}\}, \\ D_4 &= \{(x, y) \in D : 4(1-x)(1-y) - xy = 0, x < y\}, \\ D_5 &= \{(x, y) \in D : \frac{2}{3} < x = y < 1\}, \\ D_6 &= \{(\frac{2}{3}, \frac{2}{3})\}. \end{aligned}$$

Remark 4.2. By computations, we can obtain the following identities

- (a) $4(1-x)(1-y) - xy = 3 \left(y - \frac{4}{3} \right) \left(x - \frac{4}{3} \right) - \frac{4}{3}.$
- (b) $(2(1-y) - x) - (4(1-x)(1-y) - xy) = (1-y)(3x-2).$
- (c) $y(2(1-y) - x) - (2y-1)(4(1-x)(1-y) - xy) = 2(1-x)(1-y)(2-3y).$

Figure 9: D_i

Case 1 $(x, y) \in D_1$. Since $4(1-x)(1-y) - xy < 0$, g is a concave quadratic function on $[2y-1, y]$. Therefore, the function g reaches its maximum value on $[2y-1, y]$ at $y' = 2y-1$ or $y' = y$. On the other hand, since $y \geq \frac{2}{3}$, from Remark 4.1, we know that g reaches its maximum value on $[0, 2y-1]$ at $y' = \frac{y}{2}$. From (9), we obtain

$$g\left(\frac{y}{2}\right) = \frac{y(1+3x)}{4} \quad \text{and} \quad g(y) = \frac{x(1+3y)}{4}.$$

Since $x < y$, we have $g(y) < g(\frac{y}{2})$. Therefore g reaches its maximum value on $[0, y]$ at $y' = \frac{y}{2}$. Hence, we know that

$$A_{12}(x, y) = \frac{y(1+3x)}{4}.$$

Furthermore, from (8), we immediately get (as in Figure 8, let $x' = x$ and $y' = \frac{y}{2}$)

$$\tilde{\mathcal{X}}_{12}(x, y) = \left\{ \left(x, \frac{1+x}{2} \right) \right\}.$$

Case 2 $(x, y) \in D_2$. Since

$$4(1-x)(1-y) - xy > 0 \quad \text{and} \quad y \geq \frac{2}{3},$$

from Remark 4.2 (c), we can easily obtain

$$\frac{y(2(1-y)-x)}{4(1-x)(1-y)-xy} \leq 2y-1. \quad (11)$$

On the other hand, g is convex on $[2y-1, y]$, since $4(1-x)(1-y) - xy > 0$. It immediately follows from (10) and (11) that g reaches its maximum value on $[2y-1, y]$ at $y' = 2y-1$. From Remark 4.1, we know that g reaches its maximum value on $[0, y]$ at $y' = \frac{y}{2}$. Therefore, we obtain

$$A_{12}(x, y) = \frac{y(1+3x)}{4}$$

and

$$\tilde{\mathcal{X}}_{12}(x, y) = \left\{ \left(x, \frac{1+x}{2} \right) \right\}.$$

Case 3 $(x, y) \in D_3$. In this case, one can see that $4(1-x)(1-y) - xy > 0$ and $x < \frac{2}{3}$. From Remark 4.2 (b), we have

$$\frac{2(1-y) - x}{4(1-x)(1-y) - xy} < 1. \quad (12)$$

Since $y < \frac{2}{3}$, it follows from Remark 4.2 (c) and (12) that

$$2y - 1 < \frac{y(2(1-y) - x)}{4(1-x)(1-y) - xy} < y.$$

From (10), we know that g reaches its maximum value on $[2y - 1, y]$ at

$$y' = \frac{y(2(1-y) - x)}{4(1-x)(1-y) - xy}.$$

On the other hand, we have $2y - 1 < \frac{y}{2}$, since $y < \frac{2}{3}$. Hence, g reaches its maximum value on $[0, 2y - 1]$ at $y' = 2y - 1$. This immediately implies that g reaches its maximum value at

$$y' = \frac{y(2(1-y) - x)}{4(1-x)(1-y) - xy}.$$

Hence, we have

$$A_{12}(x, y) = g\left(\frac{y(2(1-y) - x)}{4(1-x)(1-y) - xy}\right) = \frac{x(1-x) + y(1-y) - xy}{4(1-x)(1-y) - xy},$$

and

$$\tilde{\mathcal{X}}_{12}(x, y) = \left\{ \left(\frac{x(2(1-x) - y)}{4(1-x)(1-y) - xy}, \frac{(2-x)(1-y) - x^2}{4(1-x)(1-y) - xy} \right) \right\}$$

Case 4 $(x, y) \in D_4$. In this case, we have $4(1-x)(1-y) - xy = 0$, and $x < y$. Hence g is a non-constant linear function on $[2y - 1, y]$. Therefore, g reaches its maximum value on $[2y - 1, y]$ at $y' = 2y - 1$ or y . By the same argument as case 1, we have

$$A_{12}(x, y) = \frac{y(1+3x)}{4}.$$

and

$$\tilde{\mathcal{X}}_{12}(x, y) = \left\{ \left(x, \frac{1+x}{2} \right) \right\}.$$

Case 5 $(x, y) \in D_5$, i.e., $\frac{2}{3} < x = y < 1$. Since $4(1-x)(1-y) - xy < 0$, g is a concave quadratic function on $[2y - 1, y]$. Therefore, the function g reaches its maximum value on $[2y - 1, y]$ at $y' = 2y - 1$ or $y' = y$. On the other hand, since $y > \frac{2}{3}$, the function g reaches its maximum value on $[0, 2y - 1]$ at $y' = \frac{y}{2}$. Note that

$$g(y) = \frac{x(1+3y)}{4} = \frac{y(1+3x)}{4} = g\left(\frac{y}{2}\right),$$

since $x = y$. Therefore g reaches its maximum value on $[0, y]$ at $y' = \frac{y}{2}$ and $y' = y$. Hence we obtain

$$A_{12}(y, y) = \frac{y(1+3y)}{4}$$

and

$$\tilde{\mathcal{X}}_{12}(y, y) = \left\{ \left(y, \frac{1+y}{2} \right), \left(\frac{y}{2}, y \right) \right\}.$$

Case 6 $(x, y) \in D_6$, i.e., $x = y = \frac{2}{3}$. From (9), we have

$$g(y') = \begin{cases} -\frac{y'^2}{2} + \frac{y'}{3} + \frac{4}{9} & 0 \leq y' \leq \frac{1}{3}, \\ \frac{1}{2} & \frac{1}{3} \leq y' \leq \frac{2}{3}. \end{cases}$$

Hence we get

$$A_{12}\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{1}{2}$$

and

$$\tilde{\mathcal{X}}_{12}\left(\frac{2}{3}, \frac{2}{3}\right) = \left\{ \left(1 - y', 1 - \frac{y'}{2} \right) : \frac{1}{3} \leq y' \leq \frac{2}{3} \right\}$$

In summary, we have

$$A_{12}(x, y) = \begin{cases} \frac{y(1+3x)}{4} & y \geq \frac{2}{3}, \\ \frac{x(1-x)+y(1-y)-xy}{4(1-x)(1-y)-xy} & y < \frac{2}{3}, \end{cases} \quad (13)$$

and

$$\tilde{\mathcal{X}}_{12}(x, y) = \begin{cases} \left\{ \left(x, \frac{1+x}{2} \right) \right\} & \frac{2}{3} \leq y, x < y, \\ \left\{ \left(\frac{x(2(1-x)-y)}{4(1-x)(1-y)-xy}, \frac{(2-x)(1-y)-x^2}{4(1-x)(1-y)-xy} \right) \right\} & y < \frac{2}{3}, \\ \left\{ \left(y, \frac{1+y}{2} \right), \left(\frac{y}{2}, y \right) \right\} & \frac{2}{3} < x = y < 1, \\ \left\{ \left(1-t, 1-\frac{t}{2} \right) : \frac{1}{3} \leq t \leq \frac{2}{3} \right\} & x = y = \frac{2}{3}. \end{cases} \quad (14)$$

where $(x, y) \in D \setminus \{(1, 1)\}$.

4.2 $A_{22}(x, y)$ and $\tilde{\mathcal{X}}_{22}(x, y)$

We first determine the case $y = 1$ (and $x \neq 1$). Let $G_2(x', y')$ be the area of $S_{\bar{f}}(x', x' + y')$ (as in Figure 10), where $x \leq x' \leq 1$ and $0 \leq y' \leq 1 - x'$, then

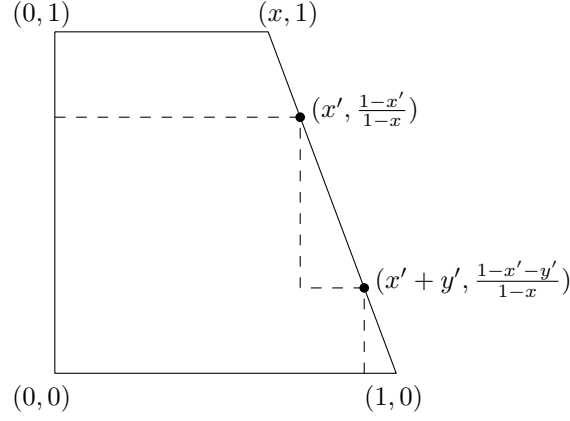
$$G_2(x', y') = \frac{1-x'^2}{2(1-x)} - \frac{1}{2(1-x)} (y'^2 + (1-x'-y')^2).$$

When x' is fixed, by Arithmetic Mean-Quadratic Mean inequality, we know that $G_2(x', y')$ reaches its maximum value at

$$y' = \frac{1-x'}{2}. \quad (15)$$

Let

$$g_2(x') = G_2\left(x', \frac{1-x'}{2}\right) = \frac{(1+3x')(1-x')}{4(1-x)}.$$

Figure 10: $S_{\bar{f}}(x', x' + y')$

Obviously, g_2 is a convex quadratic function and it reaches its maximum value on $[x, 1]$ at

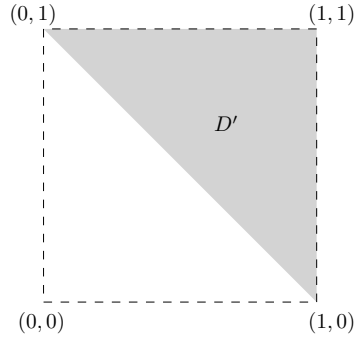
$$x' = \begin{cases} \frac{1}{3} & 0 \leq x \leq \frac{1}{3}, \\ x & \frac{1}{3} \leq x < 1. \end{cases} \quad (16)$$

and hence

$$A_{22}(x, 1) = \begin{cases} \frac{1}{3(1-x)} & 0 \leq x \leq \frac{1}{3}, \\ \frac{1+3x}{4} & \frac{1}{3} \leq x < 1. \end{cases}$$

Furthermore, From (15) and (16), we obtain (see Figure 10)

$$\tilde{\mathcal{X}}_{22}(x, 1) = \begin{cases} \left\{ \left(\frac{1}{3}, \frac{2}{3} \right) \right\} & 0 \leq x \leq \frac{1}{3}, \\ \left\{ \left(x, \frac{1+x}{2} \right) \right\} & \frac{1}{3} \leq x < 1. \end{cases}$$

Figure 11: D'

In general case, we denote $D' = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \geq 1\}$. Let $(x, y) \in D'$ and $y \neq 0$. By using affine transformation $(\tau : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \tau(s, t) = (s, \frac{t}{y}))$, one can show that

$$A_{22}(x, y) = y A_{22}(x, 1),$$

and $\tilde{\mathcal{X}}_{22}(x, y) = \tilde{\mathcal{X}}_{22}(x, 1)$. Therefore,

$$A_{22}(x, y) = \begin{cases} \frac{y}{3(1-x)} & x \leq \frac{1}{3}, \\ \frac{y(1+3x)}{4} & x \geq \frac{1}{3}. \end{cases} \quad (17)$$

and

$$\tilde{\mathcal{X}}_{22}(x, y) = \begin{cases} \left\{ \left(\frac{1}{3}, \frac{2}{3} \right) \right\} & x \leq \frac{1}{3}, \\ \left\{ \left(x, \frac{1+x}{2} \right) \right\} & x \geq \frac{1}{3}, \end{cases} \quad (18)$$

where $(x, y) \in D' \setminus \{(1, y') : 0 \leq y' \leq 1\}$.

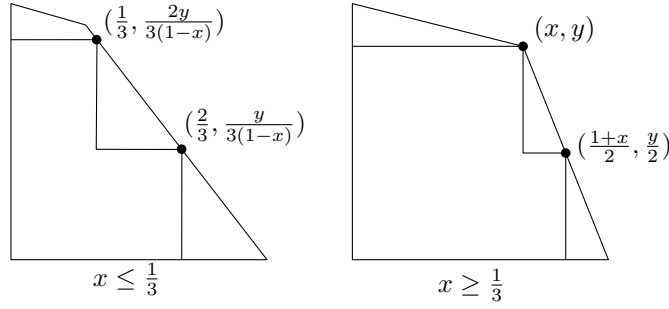


Figure 12: $\tilde{\mathcal{X}}_{22}(x, y)$

4.3 $A_{11}(x, y)$ and $\tilde{\mathcal{X}}_{11}(x, y)$

When $y = 1$, one can easily obtain

$$A_{11}(x, 1) = x,$$

and

$$\tilde{\mathcal{X}}_{11}(x, 1) = \{(x', x) : 0 \leq x' \leq x\}.$$

When $y \neq 1$. From (17), (18) and Figure 12, by swapping x with y , it is easy to see that

$$A_{11}(x, y) = \begin{cases} \frac{x}{3(1-y)} & 0 \leq y \leq \frac{1}{3}, \\ \frac{x(1+3y)}{4} & \frac{1}{3} \leq y < 1, \end{cases}$$

and

$$\tilde{\mathcal{X}}_{11}(x, y) = \begin{cases} \left\{ \left(\frac{x}{3(1-y)}, \frac{2x}{3(1-y)} \right) \right\} & 0 \leq y \leq \frac{1}{3}, \\ \left\{ \left(\frac{x}{2}, x \right) \right\} & \frac{1}{3} \leq y < 1. \end{cases}$$

Therefore, for $(x, y) \in D'$, we have

$$A_{11}(x, y) = \begin{cases} \frac{x}{3(1-y)} & y \leq \frac{1}{3}, \\ \frac{x(1+3y)}{4} & y \geq \frac{1}{3}, \end{cases} \quad (19)$$

and

$$\tilde{\mathcal{X}}_{11}(x, y) = \begin{cases} \left\{ \left(\frac{x}{3(1-y)}, \frac{2x}{3(1-y)} \right) \right\} & 0 \leq y \leq \frac{1}{3}, \\ \left\{ \left(\frac{x}{2}, x \right) \right\} & \frac{1}{3} \leq y < 1, \\ \{(t, x) : 0 \leq t \leq x\} & y = 1 \end{cases} \quad (20)$$

4.4 Determining $A_*(x, y)$

Recall that $A_*(x, y) = \max\{A_{12}(x, y), A_{11}(x, y), A_{22}(x, y)\}$. We divide D into three sets

$$\begin{aligned} B_1 &= \{(x, y) \in D : x \leq \frac{1}{3}\}, \\ B_2 &= \{(x, y) \in D : \frac{1}{3} < x \leq 1, y \geq \frac{2}{3}\}, \\ B_3 &= \{(x, y) \in D : y < \frac{2}{3}\}. \end{aligned}$$

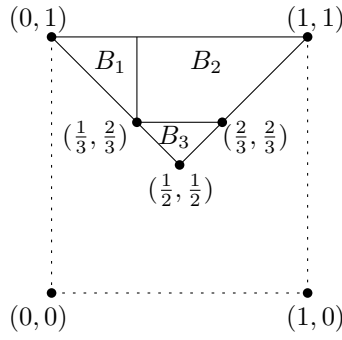


Figure 13: B_1, B_2 and B_3

Case 1 $(x, y) \in B_1$. We have

$$A_*(x, y) = \max \left\{ \frac{y(1+3x)}{4}, \frac{x(1+3y)}{4}, \frac{y}{3(1-x)} \right\}$$

Since $0 \leq x < y \leq 1$, one can show that

$$\frac{x(1+3y)}{4} < \frac{y(1+3x)}{4} \leq \frac{y}{3(1-x)}, \quad (21)$$

with $\frac{y(1+3x)}{4} = \frac{y}{3(1-x)}$ only when $x = \frac{1}{3}$. Therefore

$$A_*(x, y) = \frac{y}{3(1-x)}. \quad (22)$$

Case 2 $(x, y) \in B_2$. Note that in this case, we have

$$A_{12}(x, y) = A_{22}(x, y) = \frac{y(1+3x)}{4},$$

thus

$$A_*(x, y) = \max \left\{ \frac{y(1+3x)}{4}, \frac{x(1+3y)}{4} \right\} = \frac{y(1+3x)}{4}. \quad (23)$$

Case 3 $(x, y) \in B_3$. We have

$$A_*(x, y) = \max \left\{ \frac{x(1-x) + y(1-y) - xy}{4(1-x)(1-y) - xy}, \frac{x(1+3y)}{4}, \frac{y(1+3x)}{4} \right\}.$$

Note that

$$4(x(1-x)+y(1-y)-xy)-y(1+3x)(4(1-x)(1-y)-xy) = x(1-x)(3y-2)^2,$$

we thus have

$$\frac{x(1-x)+y(1-y)-xy}{4(1-x)(1-y)-xy} \geq \frac{y(1+3x)}{4}, \quad (24)$$

with equality only when $y = \frac{2}{3}$. Hence

$$A_*(x, y) = \frac{x(1-x)+y(1-y)-xy}{4(1-x)(1-y)-xy} \quad (25)$$

By combining (22), (23), (25) and (4), we have thus proved the second part of Theorem 2.1.

4.5 Determining $\mathcal{X}_*(x, y)$ and $\Theta(x, y)$

To determine $\mathcal{X}_*(x, y)$, we divide $D \setminus \{(1, 1)\}$ into five sets

$$\begin{aligned} E_1 &= \{(x, y) \in D : x \leq \frac{1}{3}\}, \\ E_2 &= \{(x, y) \in D : \frac{1}{3} < x < y, \frac{2}{3} \leq y\}, \\ E_3 &= \{(x, y) \in D : y < \frac{2}{3}\}, \\ E_4 &= \{(x, y) \in D : \frac{2}{3} < x = y < 1\}, \\ E_5 &= \left\{ \left(\frac{2}{3}, \frac{2}{3} \right) \right\}. \end{aligned}$$

Case 1 $(x, y) \in E_1$. From (21) and (22), we know that

$$A_{11}(x, y) < A_{12}(x, y) \leq A_{22}(x, y) = A_*(x, y),$$

Moreover, we have

$$A_{12}(x, y) = A_{22}(x, y)$$

only when $x = \frac{1}{3}$. From (14), (18) and (5), one can obtain

$$\begin{aligned} \mathcal{X}_{12}(x, y) &= \begin{cases} \tilde{\mathcal{X}}_{12}(x, y) & x = \frac{1}{3}, \\ \emptyset & \text{otherwise,} \end{cases} \\ &= \begin{cases} \left\{ \left(\frac{1}{3}, \frac{2}{3} \right) \right\} & x = \frac{1}{3}, \\ \emptyset & \text{otherwise,} \end{cases} \\ \mathcal{X}_{22}(x, y) &= \tilde{\mathcal{X}}_{22}(x, y) = \left\{ \left(\frac{1}{3}, \frac{2}{3} \right) \right\} \end{aligned}$$

and

$$\mathcal{X}_{11}(x, y) = \emptyset.$$

Hence

$$\mathcal{X}_*(x, y) = \left\{ \left(\frac{1}{3}, \frac{2}{3} \right) \right\} \quad (26)$$

Case 2 $(x, y) \in E_2$. From (13), (17), (19) and (23), one can see that

$$A_{12}(x, y) = A_{22}(x, y) = A_*(x, y)$$

and

$$A_{11}(x, y) < A_*(x, y).$$

From (14),(18) and (5), we have

$$\mathcal{X}_{12}(x, y) = \tilde{\mathcal{X}}_{12}(x, y) = \left\{ \left(x, \frac{1+x}{2} \right) \right\},$$

$$\mathcal{X}_{22}(x, y) = \tilde{\mathcal{X}}_{22}(x, y) = \left\{ \left(x, \frac{1+x}{2} \right) \right\},$$

and

$$\mathcal{X}_{11}(x, y) = \emptyset.$$

Thus,

$$\mathcal{X}_*(x, y) = \left\{ \left(x, \frac{1+x}{2} \right) \right\} \quad (27)$$

Case 3 $(x, y) \in E_3$. From (13), (17), (19), (24) and (25), we obtain

$$A_{11}(x, y) \leq A_{22}(x, y) < A_{12}(x, y) = A_*(x, y).$$

From (14),(18) and (5), we get

$$\begin{aligned} \mathcal{X}_{12}(x, y) &= \tilde{\mathcal{X}}_{12}(x, y) \\ &= \left\{ \left(\frac{x(2(1-x)-y)}{4(1-x)(1-y)-xy}, \frac{(2-x)(1-y)-x^2}{4(1-x)(1-y)-xy} \right) \right\} \end{aligned}$$

and

$$\mathcal{X}_{22}(x, y) = \mathcal{X}_{11}(x, y) = \emptyset.$$

Therefore

$$\mathcal{X}_*(x, y) = \left\{ \left(\frac{x(2(1-x)-y)}{4(1-x)(1-y)-xy}, \frac{(2-x)(1-y)-x^2}{4(1-x)(1-y)-xy} \right) \right\} \quad (28)$$

Case 4 $(x, y) \in E_4$. From (13), (17), (19) and (23), since $x = y$, we have

$$A_{12}(y, y) = A_{22}(y, y) = A_{11}(y, y) = A_*(y, y).$$

From (14),(18), (20) and (5), we obtain

$$\mathcal{X}_{12}(y, y) = \tilde{\mathcal{X}}_{12}(y, y) = \left\{ \left(y, \frac{1+y}{2} \right), \left(\frac{y}{2}, y \right) \right\},$$

$$\mathcal{X}_{22}(y, y) = \tilde{\mathcal{X}}_{22}(y, y) = \left\{ \left(y, \frac{1+y}{2} \right) \right\},$$

and

$$\mathcal{X}_{11}(y, y) = \tilde{\mathcal{X}}_{11}(y, y) = \left\{ \left(\frac{y}{2}, y \right) \right\}.$$

Hence

$$\mathcal{X}_*(y, y) = \left\{ \left(y, \frac{1+y}{2} \right), \left(\frac{y}{2}, y \right) \right\}. \quad (29)$$

Case 5 $(x, y) \in E_5$, i.e., $x = y = \frac{2}{3}$. From (22), (23) and (25), we have

$$A_{12} \left(\frac{2}{3}, \frac{2}{3} \right) = A_{22} \left(\frac{2}{3}, \frac{2}{3} \right) = A_{11} \left(\frac{2}{3}, \frac{2}{3} \right) = \frac{1}{2}.$$

From (14), (18), (20) and (5), we know that

$$\mathcal{X}_{12} \left(\frac{2}{3}, \frac{2}{3} \right) = \tilde{\mathcal{X}}_{12} \left(\frac{2}{3}, \frac{2}{3} \right) = \left\{ \left(1-t, 1-\frac{t}{2} \right) : \frac{1}{3} \leq t \leq \frac{2}{3} \right\},$$

$$\mathcal{X}_{22} \left(\frac{2}{3}, \frac{2}{3} \right) = \tilde{\mathcal{X}}_{22}(x, y) = \left\{ \left(\frac{2}{3}, \frac{5}{6} \right) \right\},$$

and

$$\mathcal{X}_{11} \left(\frac{2}{3}, \frac{2}{3} \right) = \tilde{\mathcal{X}}_{11} \left(\frac{2}{3}, \frac{2}{3} \right) = \left\{ \left(\frac{1}{3}, \frac{2}{3} \right) \right\}.$$

Hence

$$\mathcal{X}_* \left(\frac{2}{3}, \frac{2}{3} \right) = \left\{ \left(1-t, 1-\frac{t}{2} \right) : \frac{1}{3} \leq t \leq \frac{2}{3} \right\}. \quad (30)$$

By combining (3), (6), (26), (27), (28), (29), and (30), we immediately obtain the second part of Theorem 2.2. Here, we note that for the case $(x, y) = (1, 1)$, the result is trivial.

5 Determining $\delta_L(x, y)$ and $\Delta(x, y)$

Assume that $(x, y) \neq (1, 1)$. Let \bar{f} be the function defined in Section 4 (see (3)). Recall that I_1 and I_2 denote the intervals $[0, x]$ and $[x, 1]$, respectively. For $i = 1, 2$, we define

$$A^i(x, y) = \min_{x' \in I_i} |S^{\bar{f}}(x')|.$$

Let $A^*(x, y) = \min\{A^1(x, y), A^2(x, y)\}$.

In order to determine $\Delta(x, y)$, we define

$$\tilde{\mathcal{X}}^i(x, y) = \{x' : x' \in I_i, |S^{\bar{f}}(x)| = A^i(x, y)\},$$

and

$$\mathcal{X}^i(x, y) = \{x' : x' \in I_i, |S^{\bar{f}}(x')| = A^*(x, y)\},$$

where $i = 1, 2$. Obviously, for $i = 1, 2$, we have

$$\mathcal{X}^i(x, y) = \begin{cases} \tilde{\mathcal{X}}^i(x, y) & A^i(x, y) = A^*(x, y), \\ \emptyset & \text{otherwise.} \end{cases} \quad (31)$$

Let $\mathcal{X}^*(x, y) = \mathcal{X}^1(x, y) \cup \mathcal{X}^2(x, y)$. By Theorem 3.6 and (2), we know that

$$\Delta(x, y) = \{\mathcal{L}((x' - 1, 1), (x', \bar{f}(x'))) : x' \in \mathcal{X}^*(x, y)\}. \quad (32)$$

5.1 $A^*(x, y)$

Suppose that $(x, y) \in D' \setminus \{(1, 1)\}$. When $y = 1$, we can easily get

$$A^1(x, 1) = 1,$$

and

$$\tilde{\mathcal{X}}^1(x, 1) = I_1.$$

We now suppose that $y \neq 1$. Given any $x' \in I_1$. By elementary computations (see Figure 14), we have

$$|S^{\bar{f}}(x')| = 1 - \frac{(1-y)(1-x')x'}{x}.$$

Obviously, it is a concave quadratic function with respect to x' . Therefore, $|S^{\bar{f}}(x')|$ reaches its minimum value on I_1 at

$$x' = \begin{cases} \frac{1}{2} & x \geq \frac{1}{2}, \\ x & x \leq \frac{1}{2}. \end{cases}$$

Hence, we obtain

$$A^1(x, y) = \begin{cases} 1 - \frac{1-y}{4x} & x \geq \frac{1}{2}, \\ 1 - (1-x)(1-y) & x \leq \frac{1}{2}, \end{cases} \quad (33)$$

and

$$\tilde{\mathcal{X}}^1(x, y) = \begin{cases} I_1 & y = 1, \\ \{\frac{1}{2}\} & x \geq \frac{1}{2}, y < 1, \\ \{x\} & x \leq \frac{1}{2}, y < 1. \end{cases} \quad (34)$$

where $(x, y) \in D' \setminus \{(1, 1)\}$. By symmetry, we immediately get

$$A^2(x, y) = \begin{cases} 1 - \frac{1-x}{4y} & y \geq \frac{1}{2}, \\ 1 - (1-x)(1-y) & y \leq \frac{1}{2}, \end{cases} \quad (35)$$

and

$$\tilde{\mathcal{X}}^2(x, y) = \begin{cases} I_2 & x = 1, \\ \left\{1 - \frac{1-x}{2y}\right\} & y \geq \frac{1}{2}, x < 1, \\ \{x\} & y \leq \frac{1}{2}, x < 1. \end{cases} \quad (36)$$

where $(x, y) \in D' \setminus \{(1, 1)\}$.

Now we suppose that $(x, y) \in D \setminus \{(1, 1)\}$. Since $0 \leq x \leq y \leq 1$ and $x+y \geq 1$, we have

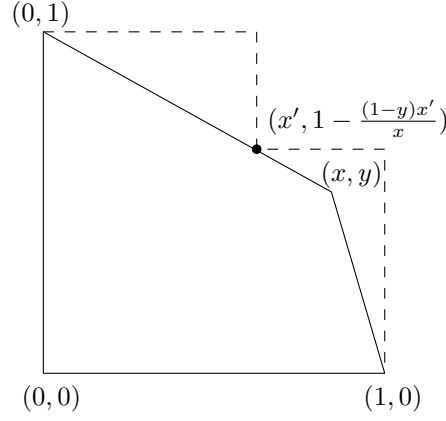
$$y(1-y) \leq x(1-x),$$

and hence

$$1 - \frac{1-x}{4y} \leq 1 - \frac{1-y}{4x}, \quad (37)$$

with equality only when $x = y$ or $x + y = 1$. On the other hand, by Arithmetic Mean-Geometric Mean inequality, we know that

$$y(1-y) \leq \frac{1}{4},$$

Figure 14: $S^{\bar{f}}(x'), x' \in I_1$

therefore

$$1 - \frac{1-x}{4y} \leq 1 - (1-x)(1-y), \quad (38)$$

with equality only when $x = 1$ or $y = \frac{1}{2}$. From (33),(35),(37) and (38), we have

$$\begin{aligned} A^*(x, y) &= \min\{A^1(x, y), A^2(x, y)\} \\ &= \min\left\{A^1(x, y), 1 - \frac{1-x}{4y}\right\} \\ &= 1 - \frac{1-x}{4y}, \end{aligned} \quad (39)$$

where $(x, y) \in D$. The first part of Theorem 2.1 immediately follows from Theorem 3.5 and (39).

5.2 $\Delta(x, y)$

We divide $D \setminus \{(1, 1)\}$ into four sets

$$\begin{aligned} F_1 &= \{(x, y) \in D : \frac{1}{2} < x = y < 1\}, \\ F_2 &= \{(x, y) \in D : x + y = 1, y > \frac{1}{2}\}, \\ F_3 &= \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}, \\ F_4 &= \{(x, y) \in D : x < y, 1 < x + y\}. \end{aligned}$$

Case 1 $(x, y) \in F_1$. From (33) and (35), since $x = y$, we have

$$A^1(y, y) = A^2(y, y) = A^*(y, y).$$

Therefore

$$\mathcal{X}^1(y, y) = \tilde{\mathcal{X}}^1(y, y) = \left\{\frac{1}{2}\right\},$$

and

$$\mathcal{X}^2(y, y) = \tilde{\mathcal{X}}^2(y, y) = \left\{ 1 - \frac{1-y}{2y} \right\}.$$

Hence

$$\mathcal{X}^*(y, y) = \left\{ \frac{1}{2}, \frac{3}{2} - \frac{1}{2y} \right\}. \quad (40)$$

Case 2 $(x, y) \in F_2$, i.e., $x + y = 1$ and $y < \frac{1}{2}$. From (33), (35) and (38), we know that

$$A^*(x, y) = A^2(x, y) = 1 - \frac{1-x}{4y} < 1 - (1-x)(1-y) = A^1(x, y).$$

Therefore

$$\mathcal{X}^1(x, y) = \emptyset,$$

and

$$\mathcal{X}^2(x, y) = \tilde{\mathcal{X}}^2(x, y) = \left\{ 1 - \frac{1-x}{2y} \right\} = \left\{ \frac{1}{2} \right\}.$$

Thus

$$\mathcal{X}^*(x, y) = \left\{ \frac{1}{2} \right\}. \quad (41)$$

Case 3 $(x, y) \in F_3$, i.e., $x = y = \frac{1}{2}$. One can see that

$$\mathcal{X}^1\left(\frac{1}{2}, \frac{1}{2}\right) = \mathcal{X}^2\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{ \frac{1}{2} \right\}.$$

Thus

$$\mathcal{X}^*\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{ \frac{1}{2} \right\}. \quad (42)$$

Case 4 $(x, y) \in F_4$. In this case, it is clear that

$$1 - \frac{1-x}{4y} < 1 - \frac{1-y}{4x},$$

and

$$1 - \frac{1-x}{4y} < 1 - (1-x)(1-y).$$

Hence, we have

$$A^*(x, y) = A^2(x, y) = 1 - \frac{1-x}{4y} < A^1(x, y).$$

Thus

$$\mathcal{X}^1(x, y) = \emptyset,$$

and

$$\mathcal{X}^2(x, y) = \tilde{\mathcal{X}}^2(x, y) = \left\{ 1 - \frac{1-x}{2y} \right\}.$$

We obtain

$$\mathcal{X}^*(x, y) = \left\{ 1 - \frac{1-x}{2y} \right\}. \quad (43)$$

By combining (3), (32), (40), (41), (42) and (43), the first part of Theorem 2.2 is complete.

6 Proof of $\delta_L(Q)\vartheta_L(Q) \geq 1$ and $\delta_L(Q) + \vartheta_L(Q) \geq 2$

To prove $\delta_L(Q)\vartheta_L(Q) \geq 1$, it suffices to show that $\delta_L(x, y)\vartheta_L(x, y) \geq 1$, for all $(x, y) \in D$. Let B_1 , B_2 and B_3 be the sets defined in Section 4.4. Suppose that $(x, y) \in D$.

Case 1 $(x, y) \in B_1$. Since $x + y \geq 1$, we have

$$x + y + 1 \geq 2 \geq \frac{4}{3(1-x)},$$

and hence

$$(x + y)^2 - 1 \geq \frac{4(x + y - 1)}{3(1-x)}.$$

Therefore

$$3(1-x)((x + y)^2 - 1) \geq 4(x + y - 1).$$

This implies that

$$3(1-x)(x + y)^2 \geq 4y - (1-x),$$

thus

$$\delta_L(x, y)\vartheta_L(x, y) = \frac{3(1-x)(x + y)^2}{4y - (1-x)} \geq 1.$$

Case 2 $(x, y) \in B_2$. Since

$$(2y - x - 1)^2 \geq 0,$$

we have

$$4y^2 - 4xy + x^2 \geq 4y - 2x - 1.$$

Hence

$$\begin{aligned} 4(x + y)^2 &= 4y^2 + 8xy + 4x^2 \\ &\geq 4y - 2x - 1 + 12xy + 3x^2 \\ &= (4y + x - 1)(1 + 3x). \end{aligned}$$

Therefore

$$\delta_L(x, y)\vartheta_L(x, y) = \frac{4(x + y)^2}{(4y - (1-x))(1 + 3x)} \geq 1.$$

Case 3 $(x, y) \in B_3$. In this case, we have

$$\delta_L(x, y)\vartheta_L(x, y) = \frac{y(x + y)^2(4(1-x)(1-y) - xy)}{(4y - (1-x))(x(1-x) + y(1-y) - xy)}.$$

To show that $\delta_L(x, y)\vartheta_L(x, y) \geq 1$, we may define

$$\pi(x, y) = y(x + y)^2(4(1-x)(1-y) - xy) - (4y - (1-x))(x(1-x) + y(1-y) - xy).$$

By computations, we obtain

$$\pi(x, y) = (x + y - 1)\pi^*(x, y),$$

where

$$\pi^*(x, y) = (3y^2 - 4y + 1)x^2 + (3y^3 - 5y^2 + 4y - 1)x - (4y^3 - 4y^2 + y).$$

We claim that $\pi^*(x, y) \geq 0$. In fact, since $\frac{1}{2} \leq y < \frac{2}{3}$, we have $3y^2 - 4y + 1 = (3y - 1)(y - 1) < 0$. Hence, π^* is a convex quadratic function with respect to x . Therefore, it suffices to show that $\pi^*(x, y) \geq 0$, when $x + y = 1$ or $x = y$. This is obvious, because if $x + y = 1$, then

$$\pi^*(x, y) = \pi^*(1 - y, y) = -6y^3 + 7y^2 - 2y = -y(3y - 2)(2y - 1) \geq 0,$$

and if $x = y$, then

$$\pi^*(x, y) = \pi^*(y, y) = y(y - 1)(3y - 2)(2y - 1) \geq 0.$$

This completes the proof of the first part of Corollary 2.4. Moreover, by Arithmetic Mean-Geometric Mean inequality, it immediately follows that

$$\delta_L(Q) + \vartheta_L(Q) \geq 2\sqrt{\delta_L(Q)\vartheta_L(Q)} \geq 2.$$

7 The proof of $\frac{1}{\delta_L(Q)} + \frac{1}{\vartheta_L(Q)} \geq 2$ and $\vartheta_L(Q) \leq 1 + \frac{5}{4}\sqrt{1 - \delta_L(Q)}$

To prove $\frac{1}{\delta_L(Q)} + \frac{1}{\vartheta_L(Q)} \geq 2$, it suffices to show that $\frac{1}{\delta_L(x, y)} + \frac{1}{\vartheta_L(x, y)} \geq 2$, for all $(x, y) \in D$. Suppose that $(x, y) \in D$.

Case 1 $(x, y) \in B_1$. We have to show that

$$\frac{4y + x - 1}{2y(x + y)} + \frac{2y}{3(x + y)(1 - x)} \geq 2,$$

or equivalently

$$\eta_1(x, y) = 3(1 - x)(4y + x - 1) + (2y)^2 - 12y(x + y)(1 - x) \geq 0.$$

By elementary computations, we can obtain

$$\eta_1(x, y) = 4(3x - 2)y^2 + 12(1 - x)^2y - 3(1 - x)^2.$$

Since $0 \leq x \leq \frac{1}{3}$, $\eta_1(x, y)$ is a convex quadratic function with respect to y . Hence, it suffices to show that $\eta_1(x, y) \geq 0$, when $x + y = 1$ or $y = 1$. This is obvious because if $x + y = 1$, then we have

$$\eta_1(x, y) = \eta_1(1 - y, y) = 4 - 3y^2 \geq 1,$$

and if $x = y$, then

$$\eta_1(x, y) = \eta_1(x, x) = (3x - 1)^2 \geq 0.$$

Case 2 $(x, y) \in B_2$. Note that

$$y^2(1 + 3x) + (4y + x - 1) - 4y(x + y) = (1 - x)(1 - y)(3y - 1) \geq 0.$$

This implies that

$$\frac{y(1 + 3x)}{2(x + y)} + \frac{4y + x - 1}{2y(x + y)} \geq 2.$$

Case 3 $(x, y) \in B_3$. We have to show that

$$\frac{2(x - x^2 + y - y^2 - xy)}{(x + y)(4 - 4x - 4y + 3xy)} + \frac{4y + x - 1}{2y(x + y)} \geq 2.$$

To do this, we define

$$\begin{aligned} \eta_3(x, y) &= 4y(x - x^2 + y - y^2 - xy) + (4y + x - 1)(4 - 4x - 4y + 3xy) \\ &\quad - 4y(x + y)(4 - 4x - 4y + 3xy) \\ &= (1 - x)((12y^2 - 15 + 4)x + 4(3y^3 - 7y^2 + 5y - 1)) \end{aligned}$$

Let

$$\eta_3^*(x, y) = (12y^2 - 15 + 4)x + 4(3y^3 - 7y^2 + 5y - 1).$$

It is clear that $\eta_3^*(x, y)$ is a linear function with respect to x . Hence, it suffices to show that $\eta_3^*(x, y) \geq 0$, when $x + y = 1$ or $x = y$. If $x + y = 1$, then

$$\eta_3^*(x, y) = \eta_3^*(1 - y, y) = y(12y + 13)(1 - y) \geq 0.$$

If $x = y$, then

$$\eta_3^*(x, y) = \eta_3^*(y, y) = 12y^3 - 43y^2 + 36y - 4.$$

By determining the second derivative, one can see that $\kappa(y) = 12y^3 - 43y^2 + 36y - 4$ is a convex function on $[\frac{1}{2}, \frac{2}{3}]$. Obviously,

$$\kappa\left(\frac{1}{2}\right) = \frac{19}{4}$$

and

$$\kappa\left(\frac{2}{3}\right) = \frac{40}{9}.$$

Therefore $\eta_3^*(y, y) = \kappa(y) \geq \frac{40}{9} > 0$.

In conclusion, we have

$$\frac{1}{\delta_L(x, y)} + \frac{1}{\vartheta_L(x, y)} \geq 2, \tag{44}$$

for all $(x, y) \in D$.

This completes the proof of the second part of Corollary 2.4. Furthermore, by elementary computations, one can show that when $\frac{2}{3} \leq u \leq 1 \leq v \leq \frac{3}{2}$ and $\frac{1}{u} + \frac{1}{v} \geq 2$, we have

$$v \leq 1 + \frac{5}{4}\sqrt{1 - u}.$$

It is known [3] that

$$\frac{2}{3} \leq \delta_L(K) \leq 1 \leq \vartheta_L(K) \leq \frac{3}{2},$$

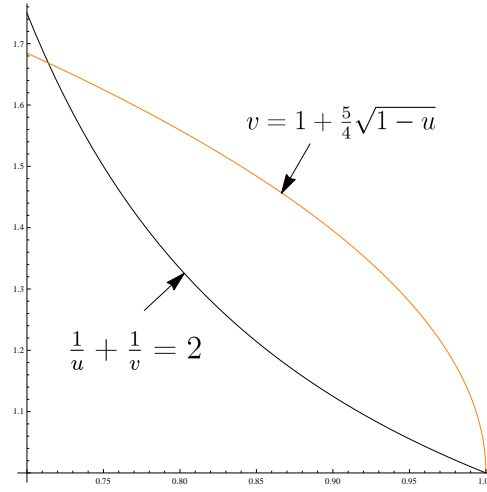
for every convex disk K . Hence

$$\frac{2}{3} \leq \delta_L(x, y) \leq 1 \leq \vartheta_L(x, y) \leq \frac{3}{2},$$

for all $(x, y) \in D$. Therefore, by (44), it immediately follows that

$$\vartheta_L(x, y) \leq 1 + \frac{5}{4}\sqrt{1 - \delta_L(x, y)},$$

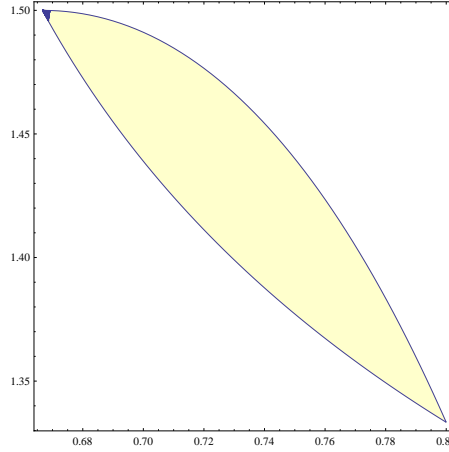
for all $(x, y) \in D$.

Figure 15: the curves $\frac{1}{u} + \frac{1}{v} = 2$ and $v = 1 + \frac{5}{4}\sqrt{1-u}$

Remark 7.1. Let \mathcal{Q} denote the collection of all convex quadrilaterals. We recall that ω_L is defined by $\omega_L(K) = (\delta_L(K), \vartheta_L(K))$ for every $K \in \mathcal{K}^2$. We can determine $\omega_L(\mathcal{Q})$. To do this, we divide \mathcal{Q} into three sets

$$\mathcal{Q}_i = \{Q : Q \text{ is affinely equivalent to } K_{x,y} \text{ for some } (x,y) \in B_i\},$$

and let $\Omega_i = \omega_L(\mathcal{Q}_i)$, where $i = 1, 2, 3$. By using computer calculations, we can obtain Ω_1 , Ω_2 and Ω_3 , as shown in Figure 16, Figure 17 and Figure 18.

Figure 16: Ω_1

Corollary 2.4 can be expressed as : $\omega_L(\mathcal{Q}) = \Omega_1 \cup \Omega_2 \cup \Omega_3$ lies between the curves $uv = 1$ and $\frac{1}{u} + \frac{1}{v} = 2$, as shown in Figure 19.

Remark 7.2. Let K_f be the convex disk defined in Section 1. We note that there exist convex disks K_f such that the inequality $\frac{1}{\delta_L(K_f)} + \frac{1}{\vartheta_L(K_f)} \geq 2$ does not hold. For example, we can let $f(x) = 1 - x^3$. By Theorem 3.5 and Theorem

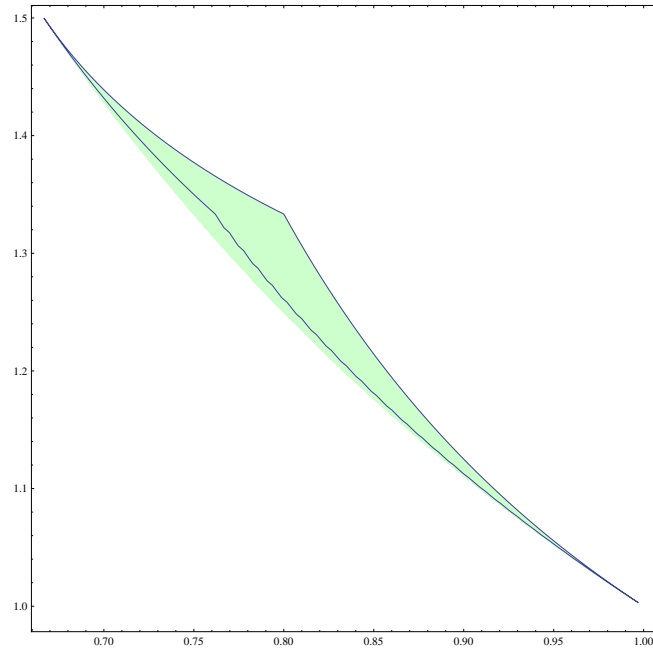


Figure 17: Ω_2

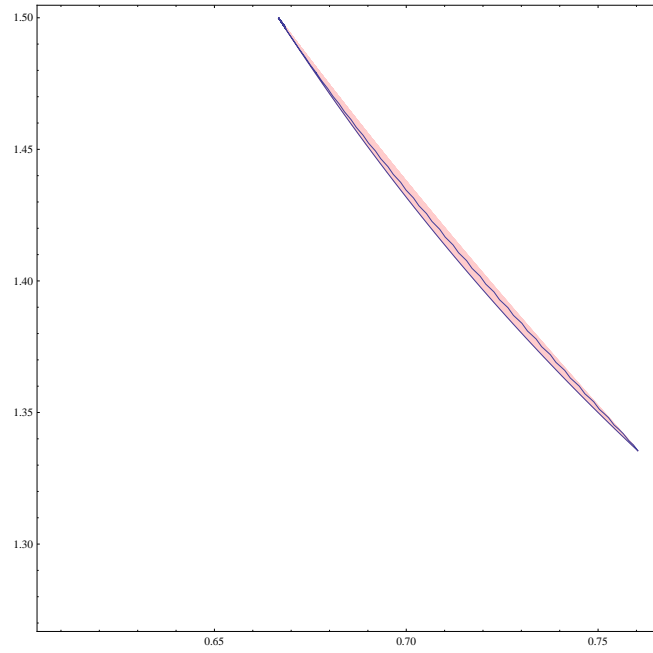
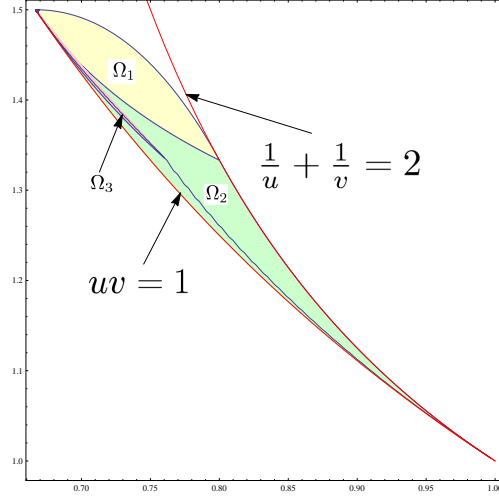


Figure 18: Ω_3

3.1, one can get

$$\delta_L(K_f) = 0.8384279476 \dots$$

Figure 19: $\omega_L(\mathcal{Q})$

and

$$\vartheta_L(K_f) = 1.282632608 \dots$$

and hence

$$\frac{1}{\delta_L(K_f)} + \frac{1}{\vartheta_L(K_f)} = 1.972354815 \dots < 2$$

8 Problems

Let K_f be the convex disk defined in Section 1. We would like to conclude with two questions

Problem 1 Does the inequality

$$\delta_L(K_f)\vartheta_L(K_f) \geq 1$$

hold for every convex disk K_f ?

Problem 2 Is it true that

$$\delta_L(K_f) + \vartheta_L(K_f) \geq 2$$

and

$$\vartheta_L(K_f) \leq 1 + \frac{5}{4}\sqrt{1 - \delta_L(K_f)}$$

for every convex disk K_f ?

Acknowledgement

I am very much indebted to Professor ChuanMing Zong for his valuable suggestions and many interesting discussions on the topic. This work is supported by 973 programs 2013CB834201 and 2011CB302401.

References

- [1] Rogers, C. A.: Packing and Covering. Cambridge university press, pp. 16–20. (1964)
- [2] Rogers, C. A.: The closest packing of convex two-dimensional domains. *Acta Math.* **86**, 309–321 (1951)
- [3] Fáry, I.: Sur la densité des réseaux de domaines convexes. *Bulletin de la S. M. F.* tome 78, 152–161 (1950)
- [4] Fejes Tóth, L.: Regular Figures. Pergamon. New York (1964)
- [5] Fejes Tóth, L.: Some packing and covering theorems. *Acta Math. Acad. Sci. Hungar.* **12**, 62–67 (1950)
- [6] Dowker, C.H.: On minimum circumscribed polygons. *Bull. Am. Math. Soc.* **50**, 120–122 (1944)
- [7] Ismailescu, D.: Inequalities between lattice packing and covering densities of centrally symmetric plane convex bodies. *Discrete Comput. Geom.***25**, 365–388 (2001)
- [8] Ismailescu, D., Kim, B.: Packing and Covering with Centrally Symmetric Convex Disks. *Discrete Comput. Geom.***51**, 495–508 (2014)
- [9] Kuperberg, W.: The set of packing and covering densities of convex disks. *Discrete Comput. Geom.***50**, 1072–1084 (2013)
- [10] Sriamorn, K., Xue, F.: On the Covering Densities of Quarter-Convex Disks, arXiv:1411.4409 (2014)